Mathematical Modeling and Numerical Simulation of the Dynamics of Flexible Structures Undergoing Large Overall Motions

J. C. SIMO†, P.I.

Division of Applied Mechanics,
Department of Mechanical Engineering,
Stanford University, Stanford, CA 94305

Final Report*

Research supported by the AFOSR under contract No. F49620-91-C-0048-P00002 with Stanford University

† Professor (deceased)

* This report was prepared by F. Armero (currently Assistant Professor, Department of Civil Engineering, UC Berkeley, 713 Davis Hall, Berkeley CA 94720) and D. D. Fox (currently at Hibbitt, Karlsson, & Sorensen, Inc., 1080 Main Street, Pawtucket, RI 02860).

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE

3. REPORT TYPE AND DATES COVERED
Final

4. TITLE AND SUBTITLE
Mathematical Modeling and Numerical Simulation of the Dynamics of Flexible Structures Undergoing Large Overall Motions (U)

5. FUNDING NUMBERS
F49620-91-C-0048-P0002
2304/A1/AS
61102F

6. AUTHOR(S)
J.C. Simo, F. Armero and D.D. Fox

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Leland Stanford Junior University
Applied Mechanics Division
Durand Building
Stanford, CA 94305-4040

8. PERFORMING ORGANIZATION REPORT NUMBER
2-DJA-826
AFOSR-TR- 95 0079

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Office of Scientific Research
AFOSR/PKA
110 Duncan Avenue, Suite B115
Bolling AFB, DC 20332-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
F49620-91-C-0048

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION / AVAILABILITY STATEMENT

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED

12b. DISTRIBUTION CODE
UL

13. ABSTRACT *(Maximum 200 words)*

This document is a final report of the research sponsored by AFOSR under contract No. F49620-91-C-0048-P0002 with Stanford University. This research effort focused on the mathematical modeling and numerical simulation of the dynamics of flexible structures undergoing large overall motions. The results reported below fall within the following areas: (i) Enhanced numerical models for shell intersections, (ii) Asymptotic methods for nonlinear shells, and (iii) Conserving time-stepping algorithms for nonlinear dynamics.

14. SUBJECT TERMS

15. NUMBER OF PAGES
14

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

Mathematical Modeling and Numerical Simulation of the Dynamics of Flexible Structures Undergoing Large Overall Motions

J. C. SIMO†, P.I.

Division of Applied Mechanics,
Department of Mechanical Engineering,
Stanford University, Stanford, CA 94305

Final report* of the research supported by the AFOSR
under contract No. F49620-91-C-0048-P00002 with Stanford University

Abstract

This document is the final report of the research sponsored by AFOSR under contract No. F49620-91-C-0048-P00002 with Stanford University. This research effort focused on the mathematical modeling and numerical simulation of the dynamics of flexible structures undergoing large overall motions. The results reported below fall within the following areas: (i) Enhanced numerical models for shell intersections, (ii) Asymptotic methods for nonlinear shells, and (iii) Conserving time–stepping algorithms for nonlinear dynamics.

1. Introduction

Realistic models of spacecrafts, space platforms and related devices invariably involve large and very flexible structures undergoing large overall motions, thus rendering inappropriate analyses based on a linearized approach. The ultimate goal of this investigation is the development of analytical and computational techniques for the mathematical modeling and numerical simulation of the nonlinear response of this type of structure undergoing such large overall motions. Central to the proposed approach is the use of nonlinear mechanical models which place no restriction on the allowable flexibility of the structure, and capture dynamic coupling effects induced by the deformation of the structure without resorting to ad–hoc simplifications. In this way, a crucial feature of the mathematical models under consideration is the rich geometric structure associated to them. The results obtained in this research project show that the exploitation of this geometric structure

† Professor (deceased)

* This report was prepared by F. Armero (currently Assistant Professor, Department of Civil Engineering, UC Berkeley, 713 Davis Hall, Berkeley CA 94720) and D. D. Fox (currently at Hibbitt, Karlsson, & Sorensen, Inc., 1080 Main Street, Pawtucket, RI 02860).

leads to very efficient numerical techniques which exhibit an excellent computational performance.

The research developed under this project has been the continuation of the research sponsored by AFOSR under grants No. DJA/AFOSR 86-0292 and No. 2DJA/AFOSR 89-0294 with Stanford University during the period 1986-1990. Accomplished results of these previous projects were: (i) The formulation and analysis of geometrically exact models, (ii) The numerical analysis of the dynamics of shell models, and (iii) The formulation of the Energy–Momentum method for the stability analysis of Hamiltonian systems. As indicated in the original proposal, the main goals of the present project are the extension and generalization of these results according to the following lines of research:

- **i.** Formulation and numerical analysis of enhanced models for junctions and shell intersections in the framework of the previously developed geometrically exact models.
- **ii.** Asymptotic methods for nonlinear shells.
- **iii.** Formulation of conserving time–stepping algorithms for nonlinear dynamics in the context of nonlinear finite element methods.

The results obtained along these three lines of research are summarized in the following section. See SIMO [1992], FOX, RAOULT & SIMO [1993], and SIMO, RIFAI & FOX [1992] for a complete description of these results in the above three respective areas.

2. Summary of the Main Accomplishments

This section presents the main results obtained in this research project. These accomplishments are grouped in the three main lines of research indicated above.

2.1. Enhanced models for junction and shell intersections

As noted in the introduction, previous research efforts under the support of the AFOSR concentrated in the formulation and numerical analysis of geometrically nonlinear shell models. See SIMO & FOX [1989], SIMO, FOX & RIFAI [1989,1990] and SIMO, FOX & RIFAI [1990] for a complete description. One of the original goals accomplished in the present research effort has been the extension and generalization of these models to accommodate situations arising in the actual design of space structures. In particular, non-smooth intersections of shells appear almost invariably in such designs. The complete mathematical characterization and numerical simulation of their mechanical response is then of major practical interest.

In recent years, there has been a significant activity in computational treatments of nonlinear shell theory. Representative examples are the recent works of PARISH [1991], where the classical degenerated solid concept is revisited in a nonlinear setting that incorporates geometrically exact rotational updates, BUCHTER & RAMM [1992] where the

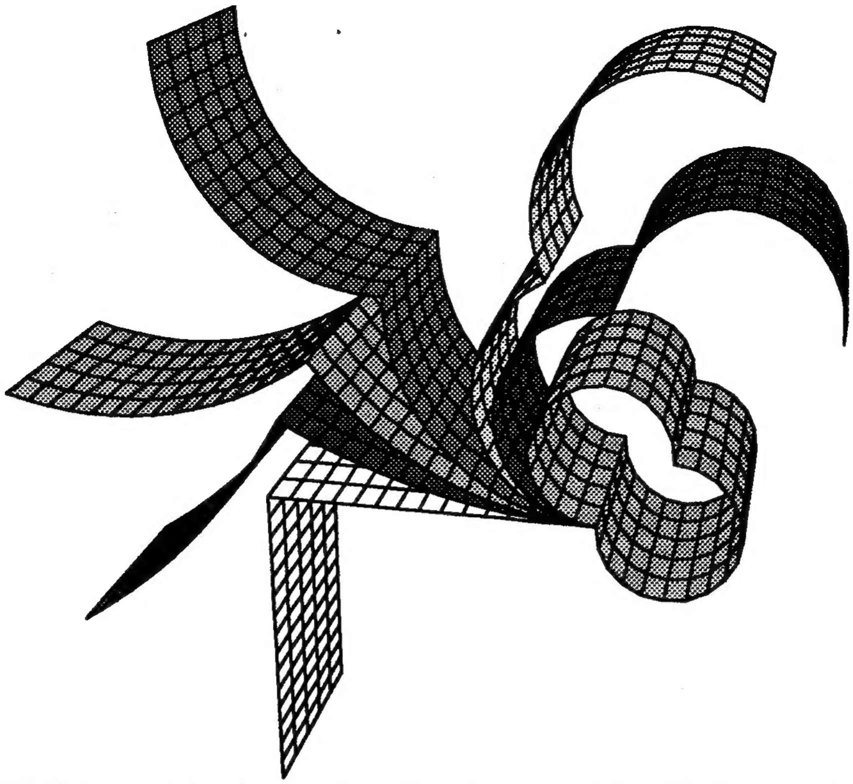

FIGURE 2.1.1 Cylindrical bending of a right angle shell, with thickness 0.02 units. Shown in the figure are different deformed shapes. Perfect agreement is found with the elementary beam solution.

relations between classical shell theories (of the type considered in this work) and the degenerated solid approach are re-examined and further explored, and SANSOUR & BUFLER [1992] where nonlinear shell theories based on the Biot formulation introduced in ATLURI [1984] are explored in a computational context. An overview of representative approaches in computational shell theory up to the end of the 80's, is contained in NOOR, BELYTSCHKO & SIMO [1990]. Inspection of the large body of literature concerned with computational shell analysis reveals that most of the approaches are restricted by the implicit assumption that the shell mid-surface is to remain smooth. The same limitation typically applies to standard expositions of nonlinear shell theory; see e.g., NAGHDI [1972]. As a result of this restriction, a unique director field is attached to each point of the mid-surface, which is parametrized by two independent (rotational) coordinates.

It has been widely accepted for a number of years now that the treatment of shell intersections using finite elements requires formulations possessing 6 DOF/Node. Current finite element methods of this type incorporate the so-called drill-rotation about the director field and often require stabilization via the addition of an artificial drill-stiffness. The recent work of PARISH [1991] is representative of this approach. Finite element methods designed to incorporate this additional rotation in the membrane field have been discussed in ALLMAN [1984], TAYLOR & SIMO [1985], HUGHES & BREZZI [1989], IBRAHIMBEGOVIC,

TAYLOR & WILSON [1990] and SIMO, FOX & HUGHES [1992] among others. It appears that non-classical theories which incorporate an independent rotation field, of the type considered by TOUPIN [1964], REISSNER [1965] and others, provide the proper continuum setting for the incorporation of drill-rotations; see e.g., FOX & SIMO [1992].

In contrast with the aforementioned works, a direct approach based on classical shell theory has been proposed in this work that circumvents the need for drill rotations. The key observation exploited in the continuum formulation of the shell intersection problem is the view of a shell intersection as a smooth curve in the reference shell mid-surface across which the director field experiences a jump discontinuity. Accordingly, two (or more) directors are assigned to each point of an intersection in the reference configuration. The additional assumption of a *rigid* shell intersection translates into the constraint which requires that the angle between the directors at each point of the intersection remain constant. Satisfaction of this constraint can be achieved only by demanding *continuity of the angular velocity* of the director field across the shell intersection (but not continuity of the director velocity itself). Since the director angular velocity at a point in a shell intersection generally possesses *three* independent components, as opposed to only *two* independent components in the case of a smooth shell mid-surface, one necessarily obtains finite element discretizations possessing 6 global DOF at points in a shell interaction. This added degree of freedom, however, is totally unrelated to the drill-rotation widely used to tackle the problem, as indicated above.

In this context, a convenient parametrization is defined in terms of a rotation matrix assigned to each node and updated iteratively. Exact nonlinear geometric updates are employed for the rotational degrees of freedom. This procedure leads to a simple modification of previously developed finite element methods. In particular, at the element level the finite element method remains unchanged from the standard formulations employing 5 DOF/Node. The element contributions for the residual vector and tangent matrix are then assembled in the corresponding global arrays, where the different number of degrees of freedom for the shell intersections are taken in account by the rotation matrix indicated above. This leads to a simple and effective computational procedure that has exhibited an excellent performance in a number of numerical simulations. Figure 2.2.1 depicts a representative numerical simulation obtained with the proposed approach consisting of the (nonlinear) rolling of a L–shape shell; see SIMO [1992] for further details.

2.2. Asymptotic methods for nonlinear shells

Plates and shells are three-dimensional continuum bodies characterized by a reference placement possessing one dimension, the thickness, which is small in comparison with the other two dimensions. This distinctive feature suggests the use of an asymptotic analysis, with the thickness regarded as a small parameter, as a means of justifying two-dimensional mathematical models of plates and shells from the full three-dimensional theory. A large

body of literature concerned with the derivation of classical plate and shell equations does in fact adopt asymptotic expansions as a central tool in the analysis. Most of the early derivations or justifications of plate and shell models via asymptotic analysis are typically restricted to the linear theory.

A reformulation and precise formalization of the asymptotic approach within the general framework of three-dimensional nonlinear elasticity is described in CIARLET & DESTUYNDER [1979] and CIARLET [1980]. There, it is shown that the von Kármán model of plates arises as the leading term in a formal asymptotic expansion of the nonlinear three-dimensional theory cast as a mixed variational problem in stresses and displacements. An alternative formulation of this approach, which uses a variational formulation solely in terms of displacements, was given by RAOULT [1988]. The asymptotic approach has been applied by these authors and their co-workers to a variety of models, including shallow shells and junctions. The lack of invariance under superposed rigid body motions (Euclidean isometries) exhibited by the (second order) models arising in this asymptotic approach, the von Kármán model in particular, is a noteworthy feature also shared by the linear theory of elasticity. This striking feature is to be contrasted with the full invariance properties under Euclidean isometries of the nonlinear three-dimensional theory which is taken as the point of departure in these asymptotic analyses.

An entirely different approach to the formulation of one and two-dimensional mathematical models for rods, plates and shells employs directly the notion of directed media introduced in COSSERAT [1909]. This approach was revitalized in the pioneering paper of ERICKSEN & TRUESDELL [1958] and encompasses classical rod models going back to EULER, CLEBCH & KIRCHHOFF (see e.g., LOVE [1932]). Comprehensive reviews of this methodology, often referred to as the *direct approach*, can be found in ANTMAN [1972] and NAGHDI [1972]. Two representative examples are the constrained two-director rod model, which incorporates finite extension, shearing, bending and twist, and the inextensible one-director shell model, which incorporates finite membrane and bending as well as transverse shear deformation. As stressed in ANTMAN [1976], a significant and often overlooked property of models arising in projection-constraint methods (intimately related to the direct approach) is the exact character of the balance laws of momentum, which can be derived directly from the three-dimensional theory. Furthermore, these models enjoy the same invariance properties under superposed Euclidean isometries as the full nonlinear three-dimensional theory.

From the above remarks, one would be tempted to conclude that the von Kármán model of plates, and not any of the full nonlinear models obtained by a direct approach, is the canonical nonlinear model that arises as the leading term in an asymptotic expansion of the three-dimensional theory. It was shown in this work that full nonlinear and properly invariant models of plates, referred to as the membrane theory and the inextensional theory, do in fact arise as leading terms in an asymptotic expansion of the three-dimensional nonlinear theory. Our approach relied on a reformulation of the asymptotic procedure

introduced in CIARLET [1980] and further developed in RAOULT [1988], cast in a form designed to preserve the full invariance properties of the three-dimensional theory. Only two assumptions were made in our formal asymptotic analysis:

i. The applied loads admit an expansion in power series of the thickness ε, after a reparametrization to an ε-independent domain is introduced.

ii. The ansatz is made that the deformations of the plate, reparametrized to the ε-independent domain, can be formally expanded in powers of ε.

We showed that, within the hierarchy of nonlinear plate models arising in the asymptotic procedure, the appearance of a specific model as the leading term is dictated only by the order of the leading non-zero term in the expansion of the loads. In particular, we showed that the nonlinear membrane theory arises if the body forces and edge loading are order 1 and the surface loading is order ε. In this classical model, the stored energy function depends only on the first fundamental form of the deformed plate mid-surface and the solutions are then characterized as minimizers of the total potential energy. No restrictions are placed by the model on the magnitude of the membrane strains.

Further, we showed that the nonlinear inextensional theory arises as the leading term in the formal asymptotic expansion for body forces and edge loading of order ε, surface loading of order ε^2 and vanishing transverse normal force (i.e., pure bending loads). In this model, the stored energy function depends only on the second fundamental form of the deformed plate mid-surface, which must be isometric to a plane since the first fundamental form is the identity. The solutions of the problem are characterized via the asymptotic procedure as minimizers of the total potential energy over the manifold of surfaces isometric to the plane. No restrictions are placed on the magnitude of the bending strains. Again it is shown that precise restrictions on the order of magnitude of the stress field are not postulated a priori (as in previous works) but are obtained as a result in the course of the asymptotic derivation.

Finally, we showed that the von Kármán model of plates arises as the leading term in the asymptotic expansion for order ε^2 in-plane and ε^3 vertical body force and edge loading, along with order ε^3 in-plane and ε^4 vertical surface loading. The preceding results, summarized in the table, provide a precise description of the position of these theories relative to the magnitude of the applied loads. In the development of these results the simplifying constitutive assumption is made of a Saint Venant-Kirchhoff material. The limitations of this model are well-known. We investigated the extension of our results to general hyperelastic constitutive models and made a preliminary outline of the steps required to perform such an analysis.

The models described above can be derived from the one-director shell model (related to the direct approach, see SIMO & FOX [1989]) by placing suitable restrictions on the director field and the shell mid-surface. In particular, the membrane theory is obtained in the absence of a director field, while the inextensional theory arises when the director

TABLE 2.2.1 Scaling of the external loading leading to different plate models.

	Body and Edge Load		Surface Load	
Plate Model	in-plane	vertical	in-plane	vertical
Membrane	1	1	ε	ε
Inextensional*	ε	ε	ε^2	ε^2
von Kármán	ε^2	ε^3	ε^3	ε^4

* For the inextensional theory, the further restriction must be made that no order ε resultant force exists (only order ε resultant moment loading).

field is identified with the normal to the deformed mid-surface which is constrained to remain isometric to the reference mid-surface (here, a plane). These results, therefore, lend considerable support to the one-director shell model as a general theory applicable to nonlinear plates/shells. Moreover, they provide a justification for the success obtained with this model in direct, large scale, numerical simulations of complex problems subjected to general loading. For a comprehensive account of numerical analysis aspects related to the implementation of the one-director nonlinear shell model we refer to SIMO, RIFAI & FOX [1990] and references therein.

2.3. Conserving time–stepping algorithms for nonlinear dynamics

Of special interest is the dynamics of very flexible shell structures undergoing finite deformations and large overall motions. In this situation there are two fundamental constants of the motion, the total linear and angular momentum maps, which are conserved by the dynamics regardless of the specific functional form assumed for the constitutive response. If one further specializes the shell model by assuming hyperelastic response and conservative loading, the mechanical system becomes Hamiltonian (see SIMO, MARSDEN & KRISHNAPRASSAD [1988]), and the total energy (i.e., the Hamiltonian) is also conserved by the dynamics. A main goal of this project is the design of time-stepping algorithms which will automatically inherit these fundamental conservation properties present in the continuum model. We showed in this work that these conservation laws are preserved in the time discretization only by a very specific class of time-stepping algorithms. Even if this class of algorithms is adopted, the conservation properties will not, in general, be preserved by the subsequent spatial discretization. A further goal of this work is the formulation of finite element interpolations which preserve the conservation laws inherited by the time-stepping algorithm. Our main results can be summarized as follows.

i. *Momentum conserving time-stepping algorithms.* A crucial aspect in the numerical simulation of the dynamics of nonlinear structural models concerns the time-intergration of the rotational dynamics which evolves in the rotation group. The difficulty is associated with the geometry of the rotation group which is not a (flat) linear space but a differentiable manifold. Recently, SIMO & WONG [1990] showed that the extension of the

Newmark formulae to the rotation group written in body coordinates, in the form suggested in SIMO & VU-QUOC [1988], along with a generalized mid-point rule approximation to Euler's angular momentum equation written in conservation form (or impulse form in the terminology of ZIENKIEWICZ, WOOD & TAYLOR [1980]), yields an algorithm which exactly preserves total angular momentum. In addition, the algorithm is frame invariant and conserves energy in the absence of external loading only for a choice of parameters leading to the *mid-point* rule.

Two nontrivial difficulties arise in the extension of the preceding class of algorithms to the dynamics of nonlinear shells. First, the coupling between the rotational and translational parts, which manifests itself in the expression for the total angular momentum map. Second, the additional S^1 symmetry present in classical shell models, which is the result of the invariance of the shell equations with respect to a one-parameter group of (drill) rotations about the director field. As discussed in SIMO & FOX [1989] and SIMO, FOX & RIFAI [1990], the enforcement of this invariance condition leads to a replacement of the rotation group by the unit sphere, since only two independent parameters are required to describe the orientation of the director field. The class of algorithms proposed in this research is then constructed as follows:

- **i.1.** Both the strong and weak forms of the Euler laws of motion for nonlinear shells are written in conservation form and approximated via a generalized mid-point rule.
- **i.2.** The Newmark formulae are extended to the unit sphere leading to a time-stepping procedure which exactly preserves frame invariance and is consistent with the classical rule of composition of angular velocities.
- **i.3.** The configurations are updated using the exponential mapping in the unit sphere, thus ensuring exact satisfaction of the unit director constraint condition.

We showed that (for Neumann boundary conditions and equilibrated loading) total angular momentum is exactly conserved only for one member of the preceding family of algorithms: *the conservation form of the mid-point rule.* The proof shows that this conservation property depends critically on adopting the steps **i.1** through **i.3** above. An identical result also holds for three-dimensional continuum mechanics. In fact, for Newmark-like algorithms, exact preservation of the conservation law of total angular momentum holds only for a mid-point rule approximation to the conservation form of the momentum equations. We remark that the accelerations play no role in such a formulation, and are obtained merely via postprocessing.

ii. *Energy conserving algorithms.* Exact energy conserving algorithms for Hamiltonian systems have received considerable attention in the literature; see e.g. GREENSPAN [1974], BAYLISS & ISAACSON [1975], LABUDDE & GREENSPAN [1976a,b] and HUGHES,

LIU & CAUGHY [1978] among others. For the linear problem, it is well-known that the mid-point rule is the only one-step method possessing this property; see BELYTSCHKO & SCHOEBERLE [1975]. We have proved a somewhat surprising generalization of this result: Given *any* stored-energy function and given an arbitrary potential for the external loading, there exists a generalized mid-point configuration, depending on the time interval $[t_n, t_{n+1}]$, for which the total energy is exactly conserved. The implementation of this result involves the actual determination of the generalized mid-point configuration as part of the algorithm. We show that this can be easily accomplished by solving an additional single scalar equation.

iii. *Algorithms incorporating numerical dissipation.* For structural dynamics, high frequency algorithmic dissipation is often regarded as a desirable property of a time-stepping algorithm. A number of modifications of the classical Newmark family of algorithms have been proposed, typically in the form of linear multi-step methods, which introduce high order dissipation and preserve second order accuracy; see e.g., BATHE & WILSON [1973], PARK [1975], WOOD, BOSSAK & ZIENKIEWICZ [1981] and BAZZI & ANDERHEGGEN [1982]. A recent review is contained in HOFF & PAHL [1988a,b]. We have presented a modification of the α-method of HILBER, HUGHES & TAYLOR [1977] which involves only a trivial modification of the preceding Newmark-like algorithms for nonlinear shells. In fact, the proposed extension of the α–method modifies the conservation form of the momentum equations with initial conditions at the beginning of the time step which does not affect the linearization of the algorithm. We showed, however, that any departure from the mid-point rule designed to introduce numerical dissipation destroys the conservation properties of the algorithm.

It is well-known (see e.g., GOUDREAU & TAYLOR [1973]), that the mid-point rule exhibits no numerical dissipation for any frequency range. In fact, for the linear problem, the algorithm possess two complex conjugate roots with unit modulus (and a zero spurious root) which bifurcate into a double real root of unit modulus at infinite sampling frequencies. According to a standard result; see e.g., GEAR [1971], the presence of a double root with unit modulus leads to a weak instability which, for the mid-point rule, can occur only for infinite sampling frequencies. As noted by CARDONA & GERARDIN [1979], infinite frequencies can occur in nonlinear dynamics if kinematic constraints are enforced via Lagrange multipliers, since they have associated massless degrees of freedom. It appears, however, that this situation is not relevant to the problem at hand since the kinematic constraint which requires the director field to be unit is enforced exactly, leading to a finite element method with five degrees of freedom per node. A similar observation applies to the dynamics of nonlinear rods (see SIMO & DOBLARE [1990]). High-frequency numerical dissipation, however, does result in added stability, since the spectral radius of the algorithm becomes strictly less that one at infinite sampling frequencies, thus preventing the appearance of weak instabilities associated with the mid-point rule. This feature is confirmed by our numerical simulations.

iv. *Finite element spatial discretization.* Arbitrary finite element interpolations of the semi-discrete shell equations do not necessarily preserve the conservation properties of the time-stepping algorithm. We have proved that for nonlinear shells the conservation properties of the algorithm are preserved by the spatial discretization if the following interpolations are adopted:

iv.1. Both the unit director constraint and the linearized director constraint conditions are enforced exactly at the nodal points of the finite element discretization. By design, these conditions are satisfied by the proposed class of time-stepping algorithms and the configuration update procedure which makes use of the exponential map.

iv.2. The discrete configuration space and the finite dimensional subspace of admissible variations are constructed using *isoparametric* finite element interpolations.

iv.3. A *row sum* approximation is used *only* for the inertia terms associated with the angular velocity of the director field.

Condition **iv.3** ensures that the S^1 invariance property of the shell equations with respect to drill rotations is also present in the discrete model and makes it possible to reduce to a five degree of freedom per node problem.

3. Concluding Remarks

We believe that this research project has led to important accomplishments in the mathematical modeling and numerical simulation of flexible structures. In particular, the aforementioned developments have led to very attractive computational methodologies for the large-scale and long-term simulation of such structures undergoing large overall motions. We would like to point out that important improvements of these results have been obtained in the development of conserving time-stepping algorithms in the subsequent research efforts under the continued support of the AFOSR. See Simo & Tarnow [1992], Simo & Tarnow [1994], Simo, Tarnow & Doblare [1994], Lewis & Simo [1994] in this respect.

References

1. Allman, D.J. [1984], " A compatible triangular element including vertex rotations for plane elasticity problems," *Computers and Structures,* **19**, 1-8.

2. Antman, S. S. [1972] "The Theory of Rods," *Handbuch der Physik*, Vol. VIa/2, Mechanics of Solids II, C. Truesdell, ed., Springer-Verlag, Berlin.

3. Antman, S. S. [1976] "Ordinary Differential Equations of Nonlinear Elasticity I: Foundations of the Theories of Non-linearly Elastic Rods and Shell," *Arch. Rational Mech. & Anal.*, **61**, Vol. 4, 307-351.

4. ATLURI, S.N. [1984], "Alternate Stress and Conjugate Strain Measures, and Mixed Variational Formulations Involving Rigid Rotations, for Computational Analyses of Finitely Deformed Solids, with Applications to Plates and Shells – I Theory," *Computers and Structures,* **18**, No.1, pp.93-116.

5. BATHE, K.J. & E.L. WILSON [1973], "Stability and accuracy analysis of direct integration methods," *Earthquake Engineering & Structural Dynamics,* **1**, 283–291.

6. BAYLISS, A., & E. ISAACSON [1975], "How to make your algorithm conservative," *Notes American Mathematical Society,* August 1975, A.594-A.595.

7. BAZZI, G. & E. ANDERHEGGEN [1982], "The ρ-Family of algorithms for time-step integration with improved numerical dissipation," *Earthquake Engineering and Structural Dynamics,* **10**, 537-550.

8. BELYTSCHKO, T. & D.F. SCHOEBERLE [1975], "On the unconditional stability of an implicit algorithm for nonlinear structural dynamics," *J. Applied Mechanics,* **42**, No. 4, 865-869.

9. BUCHTER, N. & E. RAMM [1992], "Shell theory versus degeneration - a comparison in large rotation finite element analysis," *International J. Numerical Methods in Engineering,* **43**, No.1, 39-61.

10. CARDONA, A. & M. GERARDIN [1989], "Time integration of the equations of motion in mechanism analysis," *Computers & Structures,* **33**, 801-820.

11. CIARLET, P. G. [1980] "A Justification of the von Kármán Equations," *Arch. Rational Mech. Anal.* **73**, pp. 349-389.

12. CIARLET, P. G. & DESTUYNDER, P. [1979] "A Justification of a Nonlinear Model in Plate Theory," *Comp. Meth. Appl. Mech. Engng.*, **17/18**, pp. 227-258.

13. COSSERAT, E. & F. [1909] "Théorie des Corps Déformables," appendix, pp. 953-1173, of Chwolson's *Traité de Physique*, 2nd edition, Paris.

14. ERICKSEN, J. L. & TRUESDELL, C. [1958] "Exact Theory of Stress and Strain in Rods and Shells," *Arch. Rational Mech. Anal.* **1**, pp. 295-323.

15. FOX, D.D., RAOULT, A. & SIMO, J.C. [1993] "A Justification of Nonlinear Plate Theories," *Archive for Rational Mechanics and Analysis,* **124**, 157-199.

16. FOX, D.D. & SIMO, J.C. [1992], "A Nonlinear Geometrically Exact Shell Model Incorporating Independent (Drill)Rotations, *Computer Methods in Applied Mechanics and Engineering,* **98**, 329-343.

17. GEAR, C.W. [1971], *Numerical Initial Value Problems in Differential Equations,* Prentice-Hall, Englewood Cliffs, NJ.

18. GOUDREAU, G.L. & TAYLOR, R.L. [1972], "Evaluation of numerical integration methods in elastodynamics," *Computer Methods in Applied Mechanics & Engineering*, **2**, 69–97.

19. GREENSPAN, D. [1984], "Conservative numerical methods for $\ddot{x} = f(x)$," *J. Computational Physics,* **56**, 28-41.

20. HILBER, H.M.; HUGHES, T.J.R. & TAYLOR, R.L. [1977], "Improved numerical dissipation for time integration algorithms in structural dynamics," *Earthquake Engineering and Structural Dynamics,* **5**, 283-292.

21. HOFF, C. & PAHL, P.J. [1988a], "Practical performance of the Θ_1-method and comparison with other dissipative algorithms in structural dynamics, *Computer Methods in Applied Mechanics & Engineering,* **67** 87–110.

22. HOFF, C. & PAHL, P.J. [1988b], "Development of an implicit method with numerical dissipation from a generalized single step algorithm for structural dynamics," *Computer Methods in Applied Mechanics & Engineering,* **67** 367–385.

23. HUGHES, T.J.R & BREZZI, F. [1989], "On Drill Degrees of Freedom," *Computer Methods in Applied Mechanics and Engineering*, **72**, pp. 105-121.

24. HUGHES, T.J.R.; LIU, W.K. & CAUGHY, P. [1978], "Transient finite element formulations that preserve energy," *J. Applied Mechanics,* **45**, 366-370.

25. IBRAHIMBEGOVIC, A.; TAYLOR, R.L. & WILSON, E.L. [1990], "A robust membrane quadrilateral with drilling degrees of freedom," *International Journal for Numerical Methods in Engineering,* **30**, 445-447.

26. LABUDDE, R.A., & D. GREENSPAN [1976], "Energy and momentum conserving methods of arbitrary order for the numerical integration of equations of motion. Part I," *Numer. Math.* **25**, 323-346.

27. LEWIS, D. & SIMO, J.C. [1994] "Conserving Algorithms for Nonlinear Hamiltonian Systems on Lie Groups," *Int. J. Nonlinear Science,* in press.

28. LOVE, A. L., 1932, *Mathematical Theory of Elasticity,* Dover Edition.

29. NAGHDI, P. M., 1972, "The Theory of Shells," *Handbuch der Physik*, Vol. Via/2, Mechanics of Solids II, C. Truesdell, ed., Springer-Verlag, Berlin.

30. NOOR, A.K.; BELYTSCHKO, T. & SIMO, J.C. (Editors), [1989], *Advances in Nonlinear Shell Structures,* the American Society of Mechanical Engineers, New York.

31. PARISH, H. [1991], "Investigation of a finite rotation four node assumed shell element," *International Journal for Numerical Methods in Engineering,* **31**, 127-150.

32. PARK, K.C. [1975], "Evaluating time-integration methods for nonlinear dynamic analysis," in *Finite Element Analysis of Transient Nonlinear Structural Behavior,* Eds. T. Belytschko, J.R. Osias & P.V. Marcal. Appl. Mech. Symposia Series, ASME, New York.

33. RAOULT, A. 1988, "Doctoral Dissertation, *University Pierre et Marie Curie.*

34. REISSNER, E. [1965], "A Note on Variational Principles in Elasticity," *International Journal of Solids and Structures*, **1**, pp. 93-95.

35. SANSOUR, C. & BUFLER, H. [1992], "An exact finite rotation theory, its mixed variational formulation and its finite element implementation," *International J. Numerical Methods in Engineering,* **43**, No.1, 73-117.

36. SIMO, J.C. [1992] "On a Stress Resultant Geometrically Exact Shell Model. Part VII: Shell Intersections with 5/6-DOF Finite Element Formulations," *Computer Methods in Applied Mechanics and Engineering,* **108**, 319-339.

37. SIMO, J.C. & DOBLARE, M. [1990], "Nonlinear Dynamics of Three-Dimensional Rods: Momentum Conserving Algorithms," (preprint)

38. SIMO, J.C.; FOX, D.D. & HUGHES, T.J.R. [1991] "Variational Formulations of Finite Elasticity Incorporating Independent Rotations, *Computer Methods in Applied Mechanics and Engineering,* **95**, 277-288.

39. SIMO, J.C. & FOX, D.D. [1989] "On a Stress Resultant Geometrically Exact Shell Model. Part I: Formulation and Optimal Parametrization," *Computer Methods in Applied Mechanics and Engineering,* **72**, 267-304.

40. SIMO, J. C., FOX, D. D. & RIFAI, M. S. [1989] "On a Stress Resultant Geometrically Exact Shell Model. Part II: The Linear Theory; Computational Aspects," *Computer Methods in Applied Mechanics and Engineering*, Vol. 73, pp. 53-92.

41. SIMO, J.C., FOX, D.D. & RIFAI, M.S. [1990] "On a Stress Resultant Geometrically Exact Shell Model. Part III: Computational Aspects of the Nonlinear Theory," *Computer Methods in Applied Mechanics and Engineering,* **79**, 21-70.

42. SIMO, J. C., RIFAI, M. S., & FOX, D. D. [1990] "On a Stress Resultant Geometrically Exact Shell Model. Part IV: Variable Thickness Shells with Through-the-Thickness Stretching," *Comp. Meth. Appl. Mech. Engng.* **80**, pp. 91-126.

43. SIMO, J. C., RIFAI, M. S., & FOX, D. D. [1992] "On A Stress Resultant Geometrically Exact Shell Model. Part VI: Conserving Algorithms for Nonlinear Dynamics," *International Journal for Numerical Methods in Engineering*, Vol. 34, pp. 117-164.

44. SIMO, J.C.; MARSDEN, J.E. & KRISHNAPRASSAD, J. [1987], "The Hamiltonian structure of nonlinear elasticity: The material and convective representation of solids, rods and plates," *Archive for Rational Mechanics & Analysis,* **104**, 125-183.

45. SIMO, J.C. & TARNOW, N. [1992] "The Discrete Energy–Momentum Method. Conserving Algorithms for Nonlinear Elastodynamics", *ZAMM,* **43**, 757-792.

46. SIMO, J.C. & TARNOW, N. [1994] "A New Energy and Momentum Conserving Algorithm for the Nonlinear Dynamics of Shells", *International J. Numerical Methods in Engineering,* **37**, 2527-2550.

47. SIMO, J.C.; TARNOW, N. & DOBLARE, M. [1994] "Nonlinear Dynamics of 3-D Rods: Exact Energy and Conserving Algorithms", *Int. J. Numerical Methods in Engineering,* in press.

48. SIMO J.C. & VU-QUOC, L. [1986], "A three-dimensional finite-strain rod model. Part II. Computational aspects. *Computer Methods in Applied Mechanics & Engineering,* **58**, 79–116.

49. SIMO, J.C. & WONG, K. [1990], "Unconditionally stable algorithms for the orthogonal group that exactly preserve energy and momentum," *International J. Numerical Methods in Engineering,* (to appear)

50. TAYLOR R.L. & SIMO, J.C. [1985] "Bending and Membrane Elements for the Analysis of Thick and Thin Shells," pp. 587-592 in *NUMETA 85, Numerical Methods in Engineering: Theory and Applications,* Editors: J. Middleton, and G.N. Pande, A.A. Balkema, Boston.

51. TOUPIN, R.A. [1964], "Theories of Elasticity with Couple-Stress," *Archive for Rational Mechanics and Analysis,* **17**, pp. 85-112.

52. WOOD, W.L.; BOSSAK, M. & ZIENKIEWICZ, O.C. [1981], "An alpha modification of Newmark's method," *International J. Numerical Methods in Engineering,* **15** 1562–1566.

53. ZIENKIEWICZ, O.C.; WOOD, W.L. & TAYLOR, R.L. [1980], "An alternative single–step algorithm for dynamic problems," *Earthquake Engineering and Structural Dynamics,* **8**, 31-40.